On Finding Lowest Common Ancestors: Simplification and Parallelization

by

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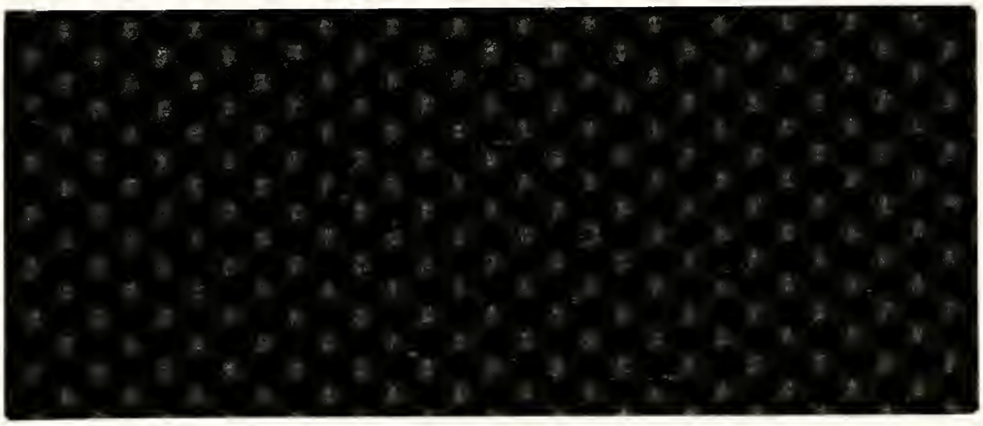

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ABSTRACT

We consider the following problem. Suppose a rooted tree T is available for preprocessing. Answer on-line queries requesting the lowest common ancestor for any pair of vertices in T. We present a linear time and space preprocessing algorithm which enables us to answer each query in $O(1)$ time, as in Harel and Tarjan [HT-84]. Our algorithm has the advantage of being simple and easily parallelizable. The resulting parallel preprocessing algorithm runs in logarithmic time using an optimal number of processors on an EREW PRAM. Each query is then answered in $O(1)$ time using a single processor.

1. Introduction

We consider the following problem. Given a rooted tree $T(V,E)$ for preprocessing, answer on-line LCA queries of the form, "Which vertex is the Lowest Common Ancestor (LCA) of x and y"? for any pair of vertices x,y in T. (Let us denote such a query $LCA(x,y)$.) We present a preprocessing algorithm which runs in linear time and linear space on the serial RAM model. (For the definition of a RAM model see, e.g., [AHU-74].) Given this preprocessing we show how to process each such LCA query in constant time.

We consider also parallelization of our algorithm. The model of parallel computation used is the exclusive-read exclusive-write (EREW) parallel random access machine (PRAM). A PRAM employs p synchronous processors all having access to a common memory. An EREW PRAM does not allow simultaneous access by more than one processor to the same memory location for either read or write purposes. See [Vi-83] for a survey of results concerning PRAMs.

Let $Seq(n)$ be the fastest known worst-case running time of a sequential algorithm, where n is the length of the input for the problem at hand. A parallel algorithm that runs in $O(Seq(n)/p)$ time using p processors is said to have *optimal speed−up* or, more simply, to be *optimal*. A primary goal in parallel computation is to design optimal algorithms that also run as fast as possible.

Our preprocessing algorithm is easily parallelized to obtain an optimal parallel preprocessing algorithm which runs in $O(\log n)$ time using $n/\log n$ processors on an EREW PRAM, where n is the number of vertices in T. Parallelizing the query processing is straightforward provided read conflicts are allowed: k queries can be processed in $O(1)$ time using k processors.

In their extensive paper [HT-84], Harel and Tarjan gave a serial algorithm for the same problem. The performance of their algorithm is the same as ours. However, our algorithm has two advantages:

(1) It is considerably simpler in both the preprocessing stage and the query processing.

(2) It leads to a simple parallel algorithm. Consider a dynamic LCA problem, in which the input is a collection of trees and edges can be added (or perhaps even removed) dynamically. [HT-84] gave an algorithm for some special case of this problem. We leave it open whether their algorithm can be simplified or whether more general versions of the dynamic LCA problem can be either simplified or improved.

Observe that using our parallel preprocessing algorithm we can process k off-line LCA queries in $O(\log n)$ time using $(n+k)/\log n$ processors provided read conflicts are allowed. This affects the performance of parallel algorithms for three problems:

(1) Given an undirected graph orient its edges so that the resulting digraph is strongly connected (if such orientation is possible) [Vi-85].

(2) Computing an open ear decomposition and st-numbering of a biconnected graph [MSV-86]. Using the new parallel connectivity and list ranking algorithms of [CV-86a] it has become possible to solve each of these problems in logarithmic time using an optimal number of processors only when $m \geq n\log n$, where n is the number of vertices and m is the number of edges in the input graph. Our off-line LCA computation enables extending the range of optimal speed-up logarithmic time parallel algorithms for these problems to sparser graphs, where $m \geq n\alpha(n,m)$ and α is inverse Ackerman's function, as in the above connectivity algorithm.

(3) Approximate string matching [LV-86]. The new parallel suffix tree construction of [LSV-86] together with the present parallel LCA computation lead to a considerable simplification of the parallel algorithm of [LV-86]. This simplification has already been described in [LSV-86].

The paper is organized as follows. Section 2 gives a high-level description of the algorithm. Section 3 describes the preprocessing stage. In Section 4 we show how to process LCA queries in T using the outcome of the preprocessing stage. Section 5 presents parallelization of our preprocessing stage.

2. High-level Description

The whole algorithm is based on the following two observations:

(1) Had our input tree been a simple path, it would have been possible to preprocess it (by way of computing the distance of each vertex from the root, as explained below) and later answer each LCA query in constant time.

(2) Had our input tree been a complete binary tree, it would have been possible to preprocess it (by way of computing its inorder numbering, as explained below) and later answer each LCA query in constant time.

The preprocessing stage assigns a number $INLABEL(v)$ to each vertex v in T. Motivated by observation (1), these numbers satisfy the following *Path Partition property*: The *INLABEL* numbers partition the tree T into paths, called *INLABEL* paths. Each *INLABEL* path consists of the vertices which have the same *INLABEL* number.
Let B be the smallest complete binary tree having at least n vertices. Our description identifies each vertex in B by its inorder number. Motivated by observation (2), the *INLABEL* numbers satisfy also the following *Inorder property*: The *INLABEL* numbers map each vertex v in T into the vertex $INLABEL(v)$ in B, such that the descendants of v are mapped into descendants of $INLABEL(v)$ in B (v is considered both a descendant and an ancestor of itself).

Section 4 describes how to process a query $LCA(x,y)$ for any pair of vertices x,y in T. The processing breaks into two cases. The simpler case is where x and y belong to the same *INLABEL* path. In the preprocessing stage we compute for each vertex v in T its distance from the root into $LEVEL(v)$. So, $LCA(x,y)$ is simply the vertex among x and y which is closer to the root. The more complicated case is where $INLABEL(x) \neq INLABEL(y)$. First, we find the LCA of $INLABEL(x)$ and $INLABEL(y)$ in the complete binary tree B, denoted by b. Let $z = LCA(x,y)$ in T. Second, we find $INLABEL(z)$. $INLABEL(z)$ is the lowest ancestor of b in B which is the *INLABEL* number of a common ancestor of x and y in T. Third, we find the lowest ancestor of x, denoted $\hat{x}$, and the lowest ancestor of y, denoted $\hat{y}$, in the path defined by $INLABEL(z)$ in T. The second and third steps use two more results of the preprocessing stage: numbers $ASCENDANT(v)$, for each vertex v, and table *HEAD*. Fourth, z is simply the vertex among $\hat{x}$ and $\hat{y}$ which is closer to the root.

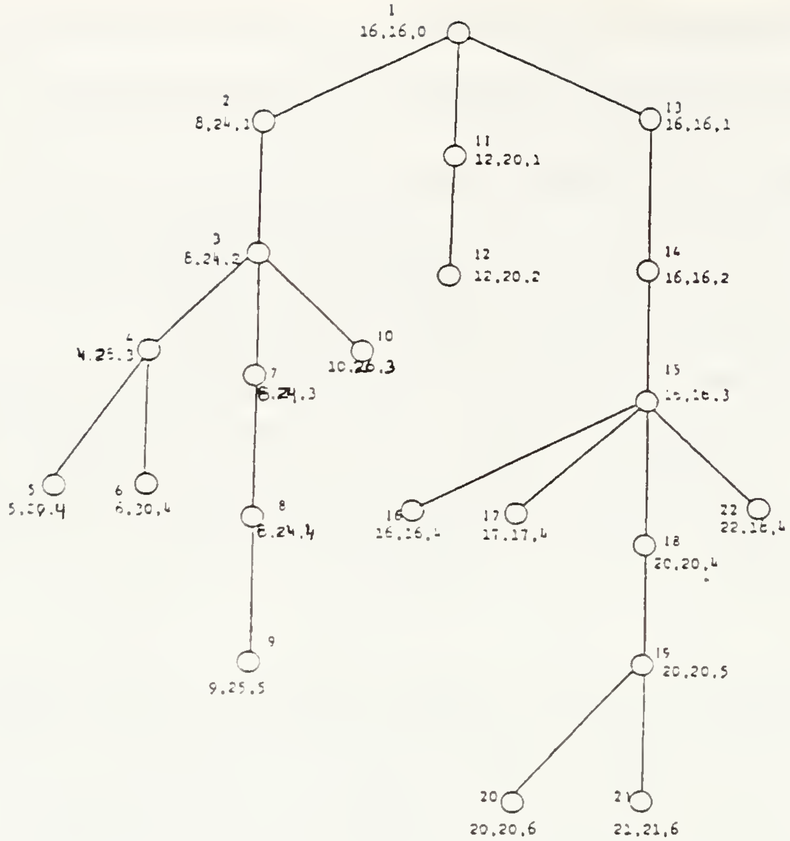

Fig. 3.1

Example. A tree with four numbers: *PREORDER, INLABEL, ASCENDANT* and *LEVEL* at each vertex.

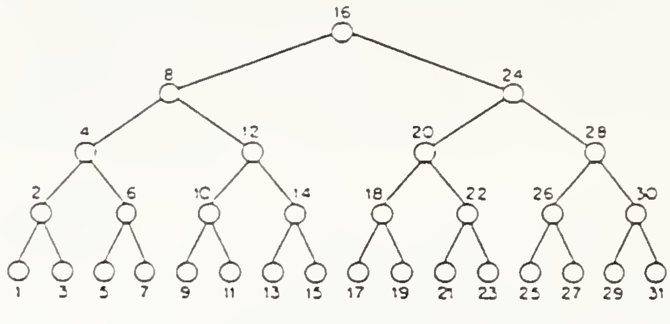

Fig. 3.2

Example. Inorder numbering of the complete binary tree with 31 vertices.

3. The Preprocessing Stage

The outcome of the preprocessing stage consists of labels which are assigned to the vertices of T and a look-up table, called *HEAD*. The label of each vertex v in T consists of three numbers: $INLABEL(v)$, $ASCENDANT(v)$ and $LEVEL(v)$.

We start with computing $INLABEL(v)$, for each vertex v in T. This is done in two steps. After a discussion of these two steps we show how to implement them.

Let $PREORDER(v)$ be the serial number of v in preorder traversal of T and $SIZE(v)$ be the number of vertices in the subtree rooted at v. Definition of preorder traversal can be found, e.g., in [AHU-74], pp. 54-55.

Step 1. Compute $PREORDER(v)$ and $SIZE(v)$.

We note that the *PREORDER* numbers of the vertices in the subtree rooted at v range between $PREORDER(v)$ and $PREORDER(v)+SIZE(v)-1$, and therefore, the closed interval $[PREORDER(v), PREORDER(v)+SIZE(v)-1]$ is called *the interval of* v.

In Step 2 we consider the binary representation of the (integer) numbers in the interval of v. We remark that throughout this paper we alternately refer to numbers and to their binary representations. No confusion will arise.

Step 2. Find the (integer) number which has the maximal number of rightmost "0" bits in the interval of v. This number is assigned to $INLABEL(v)$.

For an example of computations described in this section see Fig. 3.1.

Discussion. We show that the *INLABEL* numbers satisfy the two properties defined in the high-level description of the previous section. Observe that the intervals of the sons of v must be pairwise disjoint. Therefore, $INLABEL(v)$ belongs to the interval of at most one son of v. Denote such son by u. By the selection of the *INLABEL* numbers (Step 2), $INLABEL(u)=INLABEL(v)$ (if u exists), and for any other son w of v, $INLABEL(w) \neq INLABEL(v)$. This implies the *Partition Path property* of the *INLABEL* numbers. Let u be any descendant of v in T. Next, we show that $INLABEL(u)$ is a descendant of $INLABEL(v)$ in the complete binary tree B. (Recall that our description identifies each vertex in B by its inorder number.) Consider two vertices b and c in B. We, first, give a necessary and sufficient condition for c to be a descendant of b in B and then show that $INLABEL(u)$ and $INLABEL(v)$ satisfy this condition. Let $l = \lfloor \log n \rfloor$ and i be the number of rightmost "0" bits in b. That is, b consists of $l-i$ leftmost bits followed by a single "1" and i "0"s. The vertex c is a descendant of b if and only if

(1) the $l-i$ leftmost bits of c are the same as the $l-i$ leftmost bits of b

(2) the number of rightmost "0" bits in c is at most i. For an example of a complete binary tree and its inorder numbering see Fig. 3.2.

Let i be the number of rightmost "0" bits in $INLABEL(v)$. Since $INLABEL(u)$ belongs to the interval of v and $INLABEL(v)$ has the maximal number of rightmost "0" bits in this interval, the number of rightmost "0" bits in $INLABEL(u)$ must be at most i, and the $l-i$ leftmost bits in $INLABEL(u)$ must be the same as the $l-i$ leftmost bits in $INLABEL(v)$. This implies that $INLABEL(u)$ is the descendant of $INLABEL(v)$ in B, and, in general, the *Inorder property* of the *INLABEL* numbers.

Implementation:

Step (1) is implemented in a linear time and linear space, using preorder traversal of T. Given $PREORDER(v)$ and $SIZE(v)$, for each vertex v in T,

Step (2) is implemented in constant time per vertex in two substeps.

Step 2.1. Compute $\lfloor \log[(PREORDER(v)-1) \text{xor} (PREORDER(v)+SIZE(v)-1)] \rfloor$ into i. Let us explain this. The bitwise logical exclusive OR (denoted **xor**) of $PREORDER(v)-1$ and $PREORDER(v)+SIZE(v)-1$ assigns "1" to each bit in which $PREORDER(v)-1$ and $PREORDER(v)+SIZE(v)-1$ differ. The floor of the (base two) logarithm gives the index of the leftmost bit of difference (counting from the rightmost bit whose index is 0). Note that the bit indexed i must be "0" in $PREORDER(v)-1$ and "1" in $PREORDER(v)+SIZE(v)-1$, since the second number is larger.

Step 2.2 shows how to "compose" $INLABEL(v)$. For this, we need two observations:

(1) The $l-i+1$ leftmost bits of $INLABEL(v)$ are the same as the $l-i+1$ leftmost bits in $PREORDER(v)+SIZE(v)-1$.

(2) The i other bits in $INLABEL(v)$ are "0"s.

Step 2.2. Compute $2^i \left\lfloor \frac{PREORDER(v)+SIZE(v)-1}{2^i} \right\rfloor$ into $INLABEL(v)$. This assigns the $l-i+1$ leftmost bits in $PREORDER(v)+SIZE(v)-1$ to the $l-i+1$ leftmost bits in $INLABEL(v)$ and "0"s to the other bits of $INLABEL(v)$.

Remark: The above computation is based on *PREORDER* numbering of the vertices of T. This numbering has the property that the numbers assigned to the subtree rooted at any vertex of T provide a consecutive series of integers. In fact, any alternative numbering having this property (e.g., *POSTORDER*, *INORDER*) will produce *INLABEL* numbers which will be suitable for our preprocessing stage.

We proceed to the computation of the *ASCENDANT* numbers. The general idea is that for each vertex v, the single number $ASCENDANT(v)$ will record the *INLABEL* numbers of "*all*"the We observe that, from the viewpoint of vertex v the *INLABEL* number of each of its ancestors can be fully specified by the index of its rightmost "1". This is, since the bits which are to the left of this "1" are the same as their respective bits in $INLABEL(v)$. Like the *INLABEL* numbers, $ASCENDANT(v)$ is also an $(l+1)$-bit number. Denote the binary representation of $ASCENDANT(v)$ by the binary sequence $A_l(v),...,A_0(v)$. We set $A_i(v)=1$ only if i is the index of a rightmost "1" in the *INLABEL* number of an ancestor of v in T. To compute the *ASCENDANT* numbers, we scan the vertices of T from its root r down to its leaves (use, for instance, Breadth-First Search). We start with $ASCENDANT(r)=2^l$. Consider an internal vertex v in T and let $F(v)$ be the father of v in T. If $INLABEL(v)=INLABEL(F(v))$ then we assign $ASCENDANT(F(v))$ to $ASCENDANT(v)$, otherwise, we assign $ASCENDANT(F(v))+2^i$ to $ASCENDANT(v)$, where i is the index of the rightmost "1" in $INLABEL(v)$. It can be easily verified that i is given by $\log(INLABEL(v)-[INLABEL(v)\,\mathbf{and}\,(INLABEL(v)-1)])$, where **and** denotes bitwise logical AND.

Recall that $LEVEL(v)$, for each vertex v in T, is the distance, counting edges, of the path from v to the root r. Computation of the *LEVEL* numbers is straightforward and can be done using, e.g., Breadth-First Search.

Recall that Fig. 3.1 gives an example of the labels.

We conclude by describing how to compute the table *HEAD*. $HEAD(k)$ contains the vertex which is closest to the root in the path consisting of all vertices whose *INLABEL* number is k. $HEAD(k)$ is sometimes called the *head* of the *INLABEL* path k. Computation of the table *HEAD* is trivial. For each vertex v, such that $INLABEL(v) \neq INLABEL(F(v))$ we assign v to $HEAD(INLABEL(v))$. This, again, takes linear time and linear space.

A general implementation remark: The time bounds of both the preprocessing stage and the query processing depend on the ability to perform multiplication, division, powers of two, bitwise AND, base two discrete logarithm and bitwise exclusive OR in constant time. If these operations are not part of the machine's repertoire, look-up tables for each missing operation are prepared in linear time and linear space as part of the preprocessing stage. These tables will be used to perform the missing operations in $O(1)$ operations which are in the repertoire.

We finally note the two points in which our algorithm is similar to [HT-84]: (1) The basic observations that it is possible to answer LCA queries in simple paths and complete binary trees in constant time. (2) The idea of packing information regarding several vertices (as in the *ASCENDANT* numbers) into a single number. However, the final preprocessing stage and query processing are different.

4. Processing LCA Queries

In this section we show how to answer LCA queries using the outcome of the preprocessing stage.

Consider a query $LCA(x,y)$, for any pair of vertices x,y in T. (To illustrate the presentation the reader is referred to Fig. 3.1.) There are two cases:

(Case A) $INLABEL(x) = INLABEL(y)$. It must be that x and y are in the same *INLABEL* path. We conclude that $LCA(x,y)$ is x if $LEVEL(x) \leq LEVEL(y)$ and y otherwise.

(Case B) $INLABEL(x) \neq INLABEL(y)$. Let z be $LCA(x,y)$. We find z in four steps:

Step 1. Find b, the LCA of $INLABEL(x)$ and $INLABEL(y)$ in the complete binary tree B, as follows. Let i be the index of the rightmost "1" in b. Since b is a common ancestor of $INLABEL(x)$ and $INLABEL(y)$ in B, the $l-i$ leftmost bits in $INLABEL(x)$ and $INLABEL(y)$ must be the same as these bits in b. Since b is the *lowest* common ancestor of $INLABEL(x)$ and $INLABEL(y)$, i must be the *minimum* index such that the $l-i$ leftmost bits in $INLABEL(x)$ and $INLABEL(y)$ are the same. Hence, i is the index of the leftmost bit in which $INLABEL(x)$ and $INLABEL(y)$ differ, and b consists of the $l-i$ leftmost bits in $INLABEL(x)$ (or $INLABEL(y)$) followed by a single "1" and i "0"s.

Step 2. Find $INLABEL(z)$ (where z is $LCA(x,y)$). For this we find the index of the rightmost "1" in $INLABEL(z)$, denoted by j. Since z is a common ancestor of x and y in T, $A_j(x) = 1$ and $A_j(y) = 1$. We observe that $INLABEL(z)$ is the lowest ancestor of b in B which is also the *INLABEL* number of a common ancestor of x and y in T. Therefore, the index j must be the index of the rightmost "1" in $A_l(x),\ldots,A_i(x)$ and $A_l(y),\ldots,A_i(y)$. $INLABEL(z)$ consists of the $l-j$ leftmost bits of $INLABEL(x)$ (or $INLABEL(y)$) followed by a single "1" and j "0"s.

Step 3. Find $\hat{x}$, the lowest ancestor of x in the path defined by $INLABEL(z)$. Also, find $\hat{y}$, the lowest ancestor of y in this same path. We show how to find $\hat{x}$. $\hat{y}$ is found similarly.

If $INLABEL(x) = INLABEL(z)$ then $\hat{x} = x$ and nothing has to be done. Suppose

$INLABEL(x) \neq INLABEL(z)$. We set the following intermediate goal, as the main step towards finding $\hat{x}$: Find the son of $\hat{x}$ which is also an ancestor of x. Denote the vertex that we search by w and let k be the index of the rightmost "1" in $INLABEL(w)$. It is not difficult to verify that k is the index of the leftmost "1" in $A_{j-1}(x),\ldots,A_0(x)$. So, we find k. Clearly, $INLABEL(w)$ consists of the $l-k$ leftmost bits of $INLABEL(x)$ followed by a single "1" and k "0"s. Observe that w is the head of its *INLABEL* path (since the *INLABEL* number of its father $\hat{x}$ is different from $INLABEL(w)$). Therefore, w is $HEAD(INLABEL(w))$ and our intermediate goal is achieved. Finally, $\hat{x}$ is the father of w.

Step 4. $LCA(x,y)$ is $\hat{x}$ if $LEVEL(\hat{x}) \leq LEVEL(\hat{y})$ and $\hat{y}$ otherwise.

In the rest of this section we give additional implementation details required for the above query processing.

Step 1. To find i, the index of the rightmost "1" in b, we compute $i := \lfloor \log[INLABEL(x) \text{xor} INLABEL(y)] \rfloor$. This is similar to Step 2.1 in the *INLABEL* numbers computation of the previous section. Given i, b can be computed similarly to Step 2.2 there.

Step 2. To find j we do the following:

Step 2.1. Compute the bitwise logical AND of $ASCENDANT(x)$ and $ASCENDANT(y)$ into *COMMON*.

Step 2.2. Compute $2^i \left\lfloor \frac{COMMON}{2^i} \right\rfloor$ into $COMMON_i$. $COMMON_i$ lists all the "1"s in both $A_l(x),\ldots,A_i(x)$ and $A_l(y),\ldots,A_i(y)$.

Step 2.3. j is the index of the rightmost "1" in $COMMON_i$. To find j we compute $j := \log(COMMON_i - [COMMON_i \text{and} (COMMON_i - 1)])$, as in the *ASCENDANT* numbers computation of the previous section.

The implementation of Step 3 uses the same techniques.

5. The Parallel Preprocessing Algorithm

In this section we describe the parallel version of our preprocessing stage. It runs in $O(\log n)$ time using $n/\log n$ processors. We make the following assumption regarding the representation of the input tree T. Its $n-1$ edges are given in an array, where the incoming edges of each vertex are grouped successively. By our definition of the tree T, its edges are directed towards the root.

Computing the labels in parallel. To compute the labels of the vertices in T we apply the Euler tour technique for computing tree functions, which was given in [TV-85] and [Vi-85]. We will implement it, however, using the $O(\log n)$ time optimal parallel list ranking algorithm of [CV-86a]. This list ranking algorithm is designed for an EREW PRAM. It is based on expander graphs and its $O(\log n)$ time bound hides a constant which is not very small. We note that [CV-86b] gave recently an alternative list ranking algorithm with the same time and processor efficiencies. This alternative algorithm is designed for a PRAM which allows simultaneous access to the same memory location for both read and write purposes (called CRCW PRAM). It is simpler and its $O(\log n)$ time bound requires a small constant.

Below, we first recollect the construction required for the Euler tour technique. We then show how to use it for computing the labels. The only reason which forced us to present anew the Euler tour technique is that the computation of the *ASCENDANT* numbers has not appeared elsewhere.

Step 1. For each edge $(v \to u)$ in T we add its anti-parallel edge $(u \to v)$. Let H denote the new graph.

Since the in-degree and out-degree of each vertex in H are the same, H has an Euler path that starts and ends in r. Step 2 computes this path into the vector of pointers D, where for each edge e of H, $D(e)$ will have the successor edge of e in the Euler path.

Step 2. For each vertex v of H we do the following. (Let the outgoing edges of v be $(v \to u_0),\ldots,(v \to u_{d-1})$.) $D(u_i \to v) := (v \to u_{(i+1) \bmod d})$, for $i = 0,\ldots,d-1$. Now D has an Euler circuit. The "correction" $D(u_{d-1} \to r) := end-of-list$ (where the out-degree of r is d) gives an Euler path which starts and ends in r.

We show how to use the Euler path in order to find $PREORDER(v)$, $PREORDER(v) + SIZE(v) - 1$ and $LEVEL(v)$ for each vertex v in T.

Step 3. We assign two weights: $W_1(e)$ and $W_2(e)$ to each edge e in the Euler path as follows.

(1) $W_1(e) = 1$ if e is directed from r (that is, if e is not a tree edge) and $W_1(e) = 0$ otherwise.

(2) $W_2(e) = 1$ if e is directed from r and $W_2(e) = -1$ otherwise.

Step 4. We apply twice an optimal logarithmic time parallel list ranking algorithm to find for each e in H its (weighted) distance from the *start* of the Euler path: The first application is relative to the weights W_1 and the result is stored in $DISTANCE_1(e)$. The second application is relative to the weights W_2 and the result is stored in $DISTANCE_2(e)$. Consider a vertex $v \neq r$ and let u be its father in T. $PREORDER(v)$ is $DISTANCE_1(u \to v)+1$, $PREORDER(v)+SIZE(v)-1$ is $DISTANCE_1(v \to u)+1$, and $LEVEL(v)$ is $DISTANCE_2(u \to v)$. (These claims can be readily verified by the reader.)

Step 5. Given $PREORDER(v)$ and $PREORDER(v)+SIZE(v)-1$ for each vertex v in T we compute $INLABEL(v)$ in constant time using n processors as in the serial algorithm.

Next, we show how to use the Euler path in order to find $ASCENDANT(v)$ for each vertex v in T.

Step 6. We assign a (new) weight $W(e)$ to each edge e in the Euler path as follows. For each vertex $v \neq r$ we do the following. Let u be the father of v in T and let i be the index of the rightmost "1" in $INLABEL(v)$. If $INLABEL(v) \neq INLABEL(u)$, we assign $W(u \to v)=2^i$ and $W(v \to u)=-2^i$. The weight of all other edges is set to zero.

Step 7.
We apply again a parallel list ranking algorithm to find for each e in H its (weighted) distance from the start of the Euler path. Consider a vertex $v \neq r$ and let u be its father in T. $ASCENDANT(v)$ is the distance of the edge $(u \to v)$ plus 2^l. Clearly, $ASCENDANT(r)=2^l$.

We note that, given the labels, the table *HEAD* can be computed in constant time using n processors.

Complexity. Each of steps 4 and 7 needs $n/\log n$ processors and $O(\log n)$ time. Each of steps 1,2,3,5,6 and the computation of *HEAD* needs n processors and $O(1)$ time and can be readily simulated by $n/\log n$ processors in $O(\log n)$ time. Thus, the parallel preprocessing stage can be done in a total of $O(\log n)$ time using $n/\log n$ processors.

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